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We propose a selfconsistent quantum mechanical approach to study the dynamics of a two-level system subject to random time evolution. This randomness gives rise to competing effects between dissipative and non-dissipative decoherence with a consequent slow down of the atomic decay rate.

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I. INTRODUCTION

Time in Quantum Mechanics plays a rather ambiguous role. Usually, it enters as a continuous parameter, but the principle of relativity suggests a parallelism between position-momentum and time-energy. Thus, there was the attempt to quantize the time [1]. Otherwise, it has been considered as a discrete variable [2,3]. This approach implies a modification of the Schrödinger equation providing also an explanation of the nonappearance of macroscopically distinguishable states in terms of non-dissipative decoherence.

More recently, a generalization of the Liouville-Von Neumann equation [4] was developed without any specific statistical assumption [5]. It results an expression for the density operator which, a posteriori, can be interpreted as if the evolution time is not a fixed parameter, but a stochastic variable whose distribution is a Γ -distribution. It also provides a model for quantum measurement. Furthermore, the characteristic time appearing in such approach could be related to the time-energy uncertainty relation [5,6]. This approach is also useful to explain decoherence whenever the environmental degrees of freedom responsible for decoherence are not easily recognizable [7].

Here, we shall adopt this approach for a dissipative two-level system. Then, we shall show how decoherence effects compete in this framework in different regimes. In particular, we shall give an example of how statistical quantum theory can describe a Zeno-like effect, i.e. frozen decay, which is generally thought to arise from the state reduction caused by the measurements [8].

II. STANDARD APPROACH

Usually, the spontaneous decay of a two-level atom is described by means of a master equation derived by assuming the system interacting with an environment [9].

Consider a atom, with two relevant levels $\{|g\rangle, |e\rangle\}$ and lowering operator $\sigma = |g\rangle\langle e|$. Let ω be the energy difference between the two levels ($\hbar = 1$), and let γ be the decay rate. Then, the master equation we are concerning is derived by considering an interaction of the type $H \propto (\sigma\Gamma^\dagger + \sigma^\dagger\Gamma)$ where Γ is a bath operator. Under the Born-Markov approximation one arrives to [10]

$$\dot{\rho} = \mathcal{L}\rho, \quad (1)$$

with the Liouvillian superoperator

$$\mathcal{L}\rho = -i\omega[\sigma_z, \rho] + \frac{\gamma}{2}(2\sigma\rho\sigma^\dagger - \sigma^\dagger\sigma\rho - \rho\sigma^\dagger\sigma), \quad (2)$$

where, we choose to define the inversion operator as $\sigma_z = \sigma^\dagger\sigma - \sigma\sigma^\dagger$, and the quadrature operators as $\sigma_x = \sigma + \sigma^\dagger$ and $\sigma_y = i\sigma - i\sigma^\dagger$.

Dephasing processes, if necessary, are introduced in the same way. These processes do not change the population of the two-level atom but do cause a phase randomization of the atomic dipole. Then, they can be modeled by considering the inversion σ_z to be coupled to environment by $H \propto \sigma_z(\Gamma + \Gamma^\dagger)$. The master equation (1) now becomes

$$\dot{\rho} = -i\omega[\sigma_z, \rho] + \frac{\gamma}{2}(2\sigma\rho\sigma^\dagger - \sigma^\dagger\sigma\rho - \rho\sigma^\dagger\sigma) - \kappa[\sigma_z, [\sigma_z, \rho]], \quad (3)$$

where κ represents the phase decaying rate.

From the master Eq.(3) it is easy to derive the following dynamical equations

$$\text{Tr}\{\dot{\rho}\sigma_z\} = -\gamma\text{Tr}\{\rho\sigma_z\} - \gamma, \quad (4)$$

$$\text{Tr}\{\dot{\rho}\sigma\} = -\left[i\omega + \left(\frac{\gamma}{2} + \kappa\right)\right]\text{Tr}\{\rho\sigma\}. \quad (5)$$

The solutions read

$$\text{Tr}\{\rho(t)\sigma_z\} = \text{Tr}\{\rho(0)\sigma_z\}\exp(-\gamma t) + [\exp(-\gamma t) - 1], \quad (6)$$

$$\text{Tr}\{\rho(t)\sigma\} = \text{Tr}\{\rho(0)\sigma\}\exp\left[-i\omega t - \left(\frac{\gamma}{2} + \kappa\right)t\right]. \quad (7)$$

We may see that the equation of motion for the inversion is unchanged with respect to the dissipative case, but the polarization decay rate is increased above the spontaneous emission result.

Nevertheless, decoherence is not always necessarily due to the entanglement with an environment but it may be due, especially the non-dissipative one, to the fluctuations of some classical parameters or internal variable of the system. Or it might have an “intrinsic” character [11]. Hence, we shall present a more general approach to non-dissipative decoherence for a two-level system.

Quantum mechanics is a statistical theory whose elements are ensembles of quantum systems, or ensembles of measurements on the same quantum system. This led, long time ago, to the introduction of the density operator [4]. Along this line, we cannot state a priori that time is uniquely determined within the ensemble. Rather, it would be more reasonable to give a statistical interpretation of the time variable too. Then, following Ref. [5], the evolution of a system is averaged on a suitable probability distribution $P(t, t')$ where t' represents all possible times within the ensemble. Let $\rho(0)$ be the initial state, then the evolved state would be

$$\bar{\rho}(t) = \int_0^\infty dt' P(t, t') \rho(t'), \quad (8)$$

where $\rho(t') = \exp\{-i\mathcal{L}t'\}\rho(0)$ is the solution of the Liouville-Von Neumann equation [4].

One can write as well

$$\bar{\rho}(t) = \mathcal{V}(t)\rho(0), \quad (9)$$

where the superoperator \mathcal{V} is given by

$$\mathcal{V}(t) = \int_0^\infty dt' P(t, t') e^{-i\mathcal{L}t'}. \quad (10)$$

In Ref. [5], the function $P(t, t')$ has been determined to satisfy the following conditions: i) $\bar{\rho}(t)$ must be a density operator, i.e. it must be self-adjoint, positive-definite, and with unit-trace. This leads to the condition that $P(t, t')$ must be non-negative and normalized, i.e. a probability density in t' , so that Eq.(8) is a completely positive map; ii) $\mathcal{V}(t)$ satisfies the semigroup property $\mathcal{V}(t_1 + t_2) = \mathcal{V}(t_1)\mathcal{V}(t_2)$, with $t_1, t_2 \geq 0$. These requirements are satisfied by

$$\mathcal{V}(t) = \frac{1}{(1 + i\mathcal{L}\tau)^{t/\tau}}, \quad (11)$$

and

$$P(t, t') = \frac{1}{\tau} \frac{e^{-t'/\tau}}{\Gamma(t/\tau)} \left(\frac{t'}{\tau}\right)^{(t/\tau)-1}, \quad (12)$$

where the parameter τ naturally appears as a scaling time. Notice that the evolution superoperator (11) only depends on t , and parametrically on τ as in “non extensive” generalization of Liouville equation [12]. Indeed, t' comes out when a statistical interpretation is employed. Expression (12) is the so-called Γ -distribution function, well known in line theory [13]. The meaning of the parameter τ can be understood by considering the mean $\langle t' \rangle = t$, and the variance $\langle t'^2 \rangle - \langle t' \rangle^2 = \tau t$. Hence, τ rules the strength of time fluctuations, or, otherwise, the characteristic correlation time of fluctuations.

When $\tau \rightarrow 0$, $P(t, t') \rightarrow \delta(t - t')$ so that $\bar{\rho}(t) \equiv \rho(t)$ and $\mathcal{V}(t) = \exp\{-i\mathcal{L}t\}$ is the usual evolution.

It is worth noting that the behavior of the distribution (12) strongly depends on the regime we consider. In fact, for $t \ll \tau$ we have an exponential behavior, while for $t \gg \tau$ a Gaussian-like shape. The case $t = \tau$ represents the border between these two behaviors. All that is illustrated in Fig.(1).

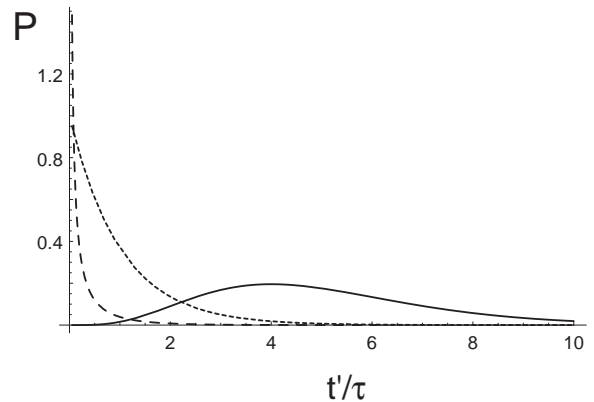


FIG. 1. Probability distribution (12) as function of dimensionless variable t'/τ . The dashed line refers to $t/\tau = 0.1$, the dotted line to $t = \tau$, and the solid line to $t/\tau = 5$.

The phase diffusion aspect of the present approach can also be seen in the evolution equation for the averaged density matrix $\bar{\rho}(t)$. Indeed, by differentiating with respect to time Eq.(9) and using (11) one gets the following master equation for $\bar{\rho}(t)$

$$\dot{\bar{\rho}}(t) = -\frac{1}{\tau} \log(1 + i\mathcal{L}\tau) \bar{\rho}(t). \quad (13)$$

Once $\mathcal{L}\rho \equiv [H, \rho]$, the evolution operator $\mathcal{V}(t)$ describes a decay of the off diagonal matrix elements in the energy representation, whereas the diagonal matrix elements remain constants, i.e. the energy is still a constant of motion. In fact, in the energy eigenbasis, Eqs.(9) and (11) yield

$$\bar{\rho}_{n,m}(t) = \exp(-\kappa_{n,m}t) \exp(-i\nu_{n,m}t) \rho_{n,m}(0), \quad (14)$$

where

$$\kappa_{n,m} = \frac{1}{2\tau} \log(1 + \omega_{n,m}^2 \tau^2), \quad (15)$$

$$\nu_{n,m} = \frac{1}{\tau} \arctan(\omega_{n,m}\tau), \quad (16)$$

with $\omega_{n,m}$ the energy difference. One can recognize in Eq.(14), beside the exponential decay, a frequency shift of every oscillating term. This can be also used as a model for Quantum Nondemolition Measurement [10]. In fact, in standard quantum measurement theory each measurement results in an instantaneous reduction of the wave function onto an eigenstate corresponding to the particular detected eigenvalue of the observable being measured. Non-selective measurements destroy the phase relation between different eigenstates and reduce

the state of the system to a statistical mixture where the non-diagonal elements of the corresponding density matrix vanish. Therefore, all random dephasing events, i.e all processes that provide for a rapid quantum mechanical phase destruction but leave the diagonal elements of the density matrix unchanged, cause the same dynamical effect on the evolution of the system like genuine quantum-nondemolition measurements.

However, it would also be possible to consider the Liouvillian (2) in Eq.(13), and therefore the competition between two types of decoherence. This is what we are going to study in the following.

IV. SYSTEM DYNAMICS WITH RANDOM TIME EVOLUTION

If τ is small enough, one can expand the logarithm in (13) up to second order in τ , and by using the Liouvillian (2), we obtain

$$\begin{aligned} \dot{\bar{\rho}}(t) = & -i\omega [\sigma_z, \bar{\rho}(t)] \\ & + \frac{\gamma}{2} (2\sigma\bar{\rho}(t)\sigma^\dagger - \sigma^\dagger\sigma\bar{\rho}(t) - \bar{\rho}(t)\sigma^\dagger\sigma) \\ & - \frac{\tau}{2}\omega^2 [\sigma_z, [\sigma_z, \bar{\rho}(t)]] , \end{aligned} \quad (17)$$

where we have used $\omega \gg \gamma$ and $\tau \ll \gamma^{-1}$, that is, the dissipation takes place on a time scale much larger than the time fluctuations.

Eq.(17) practically coincides with Eq.(3) provided to identify $\tau\omega^2/2$ with κ . Nonetheless, the present approach is different from the usual master equation approach, in the sense that it is model independent and without specific statistical assumptions.

For a generic value of τ , it is not possible to extract an explicit form of master equation from Eq.(13). Nevertheless, the physics of the system can be understood by simply averaging the quantities of interest over the distribution (12). For instance, from Eqs.(13) and (2), we get

$$\begin{aligned} \text{Tr} \{ \bar{\rho}(t) \sigma_z \} = & \text{Tr} \{ \bar{\rho}(0) \sigma_z \} \exp \left[-\frac{t}{\tau} \log(1 + \gamma\tau) \right] \\ & + \left\{ \exp \left[-\frac{t}{\tau} \log(1 + \gamma\tau) \right] - 1 \right\} . \end{aligned} \quad (18)$$

Equation (18) in the limit $\gamma\tau \ll 1$ reduces to the usual decay described by Eq.(6). More generally, the decay rate results modified. In particular, for $\gamma\tau \gg 1$ it would be possible to inhibit the dissipative effects through the nondissipative ones. The frozen dynamics due to increasing values of $\gamma\tau$ is shown in Fig.(2). This situation comes out as consequence of the transition from the Gaussian to the exponential behavior of the probability distribution (12) (see Fig.1).

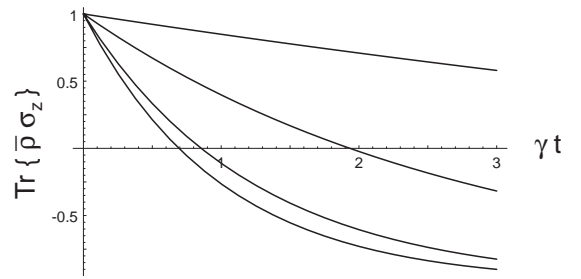


FIG. 2. Population inversion as function of the scaled (dimensionless) time γt , for different values of $\gamma\tau$. From bottom to top $\gamma\tau = 0, 0.5, 5, 50$. The system has been considered initially in the upper level.

The freezing effect on the system dynamics remind us the quantum Zeno effect [8]. Its usual description rests on the suppression of the unitary Hamiltonian evolution of a quantum system due to intermittent measurements in rapid succession. Due to the wave function collapses, in the limit of continuous measurements, the evolution is completely inhibited, and the system is frozen in its initial state. Here, instead, the effect entirely arise from quantum statistical properties.

While in the usual quantum Zeno effect the essential requirement is that the measurements of the system state, which cause the interruption, be more closely spaced in time than the reservoir correlation (memory) time, in our case the correlation time of fluctuations should exceed the typical decay time.

Essentially, we may claim that in the limit $\gamma\tau \gg 1$ one type of decoherence prevents the other. In fact, we may think that dissipative decay process takes place through the energy channels determined by the system-environment interaction. However, the time evolution fluctuations make these channels completely fuzzy, thus preventing the decay.

The above results can be easily extended to the case of driven two-level system.

V. CONCLUSIONS

In conclusion, we have presented a simple model able to explain different aspects of decoherence in a two-level system. The used formalism predicts, for large time fluctuations, a novel Zeno-like effect without invoking the abstruse concept of wavefunction collapse [14].

The generality of the presented approach suggests in some way the possibility that the parameter τ (even though system-dependent) might have a lower nonzero limit, related e.g. to the time energy uncertainty relation [5], or to the finite extension of the spatial wavefunction [15], or even to gravitational effects [16]. However, even if such “intrinsic” decoherence effects emerge, the value of τ would be very small. Nevertheless, one can think as well to introduce the above statistical properties by hand from the outside. For instance, one can use dephasing processes through a noisy driving field as

envisaged in Ref. [17]. Otherwise, one could think at a sequence of measurements as jump-like processes, randomly distributed in time [18]. In such a cases the statistics, hence the parameter τ , would be controlled by the experimenter. Thus, it would be an interesting challenge to arrange an experimental set up where the conditions for above Zeno-like effect are achieved.

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